Line SIAC Filtering: Effective Rotation for Multidimensional Fields

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Dedicated to Prof. Chi-Wang Shu in honor of his 60th birthday

Introduction

SIAC filtering is a post-processing technique designed to increase the smoothness and extract the hidden "superconvergence" of numerical solutions obtained through a Discontinuous Galerkin (DG) method [1]. It consists of convolving a B-Spline kernel at a particular point with the DG solution at final time:

$$u^\star(\bar{x}, T) = \frac{1}{H}\int \underbrace{K_H^{(2k+1,k+1)}(x)}_{\text{SIAC kernel}}\underbrace{u_h(x,T)}_{\text{DG solution}}\,dx.$$

Since the DG solution is continuous only inside the elements, the error exhibits high frequency oscillations. Applying a SIAC filter, the oscillations can be removed:

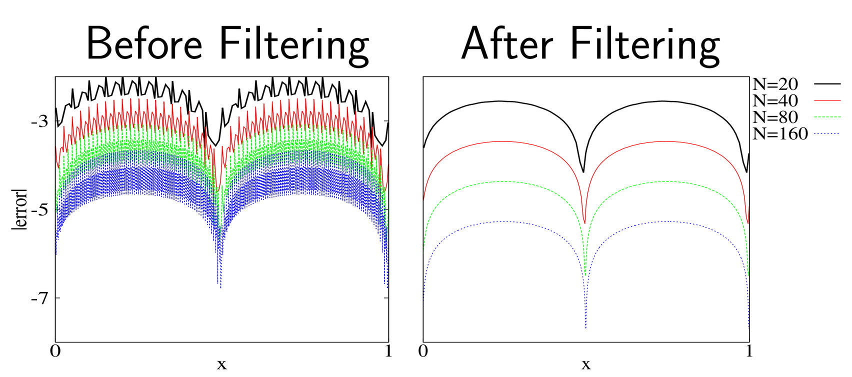


Figure: $u_t + u_x = 0$, $u(x,0) = \sin(2\pi x)$, $T = 12.5$

Traditionally, the applications of these filters in multidimension have employed a tensor product kernel. However, this structure results in a filter that grows in support as the field dimension increases, becoming computationally expensive. Here, we present Line SIAC filters: a new and computationally efficient approach for post-processing multidimensional data by transforming the integral of the convolution into a line integral.

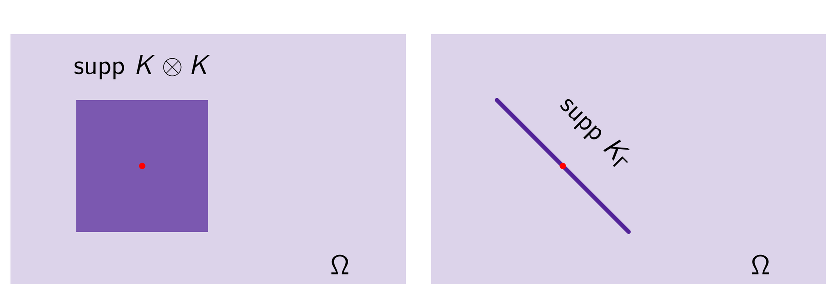


Figure: Integration regions for post-processing a particular point using tensor product and line filters.

This affords great advantages towards the applications of these filters since the simulation times become significantly shorter and the complexity of the algorithm design reduces to a one-dimensional problem.

Line SIAC Filter

Given a rotation line $\Gamma(t) = t(\cos\theta, \sin\theta) + (\overline{x,y})$, θ fixed, the filtering convolution is defined by:

$$u^\star(\overline{x,y}) = \frac{1}{H}\int_{-\infty}^{\infty} K_\Gamma^{(2k+1,k+1)}\left(\frac{t}{H}\right) u_h(\Gamma(t))dt.$$

The **kernel** is defined in the usual way:

$$K^{(2k+1,k+1)}(\cdot) = \sum_{\gamma=-k}^{k} c_\gamma \psi^{(k+1)}(\cdot - \gamma),$$

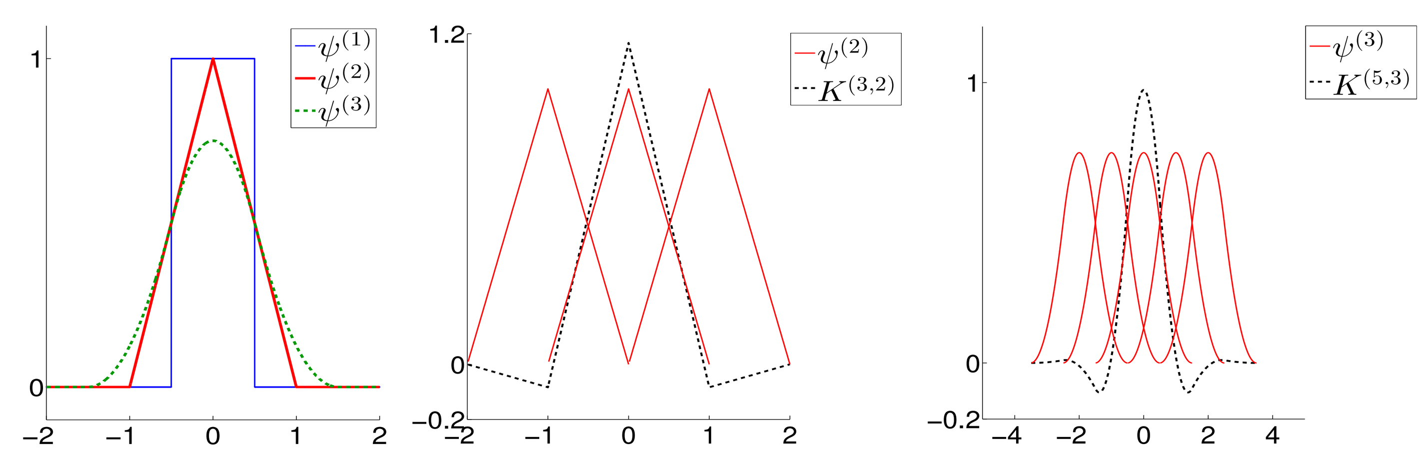
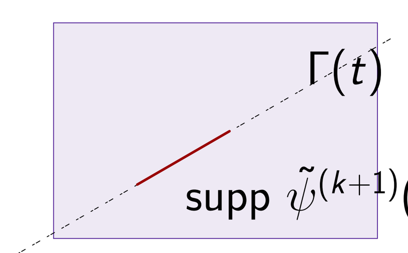


Figure: B-Splines (left) and two symmetric SIAC kernels (centre and right).

but the B-Splines are defined along the rotation line:

$$\tilde{\psi}_\theta^{(k+1)}(x,y) = \begin{cases} \psi^{(k+1)}\left(\Gamma^{-1}(x,y)\right) & \text{if } (x,y) \in \Gamma(t) \\ 0 & \text{otherwise.}\end{cases}$$



$$t \in \left[-\frac{k+1}{2}, \frac{k+1}{2}\right]$$

Theoretical Results

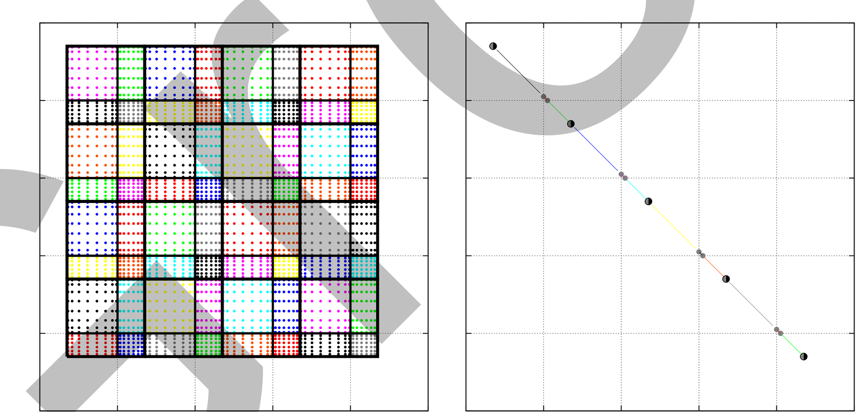
For linear hyperbolic problems and uniform meshes, we can show that

$$\|u - K_{\Gamma,H}^{(2k+1,k+1)} \star u_h\|_{L^2} \leq Ch^{2k+1},$$

where k is the degree of the DG solution [2].

Computational Advantages

Reducing the dimension implies less number of integration regions. This number matches the number of quadrature sums. A tensor product filter requires n^2 quadrature sums!.

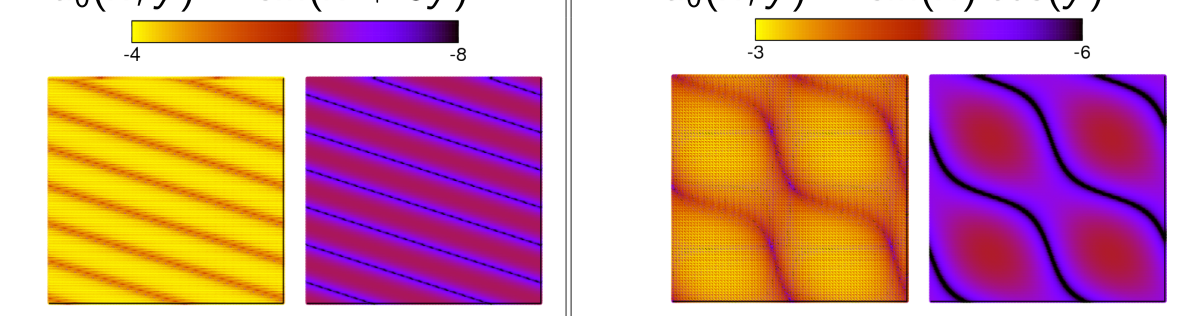


Kernel	Integrals		Quadrature Sums	
	$K \otimes K$	K_Γ	$K \otimes K$	K_Γ
$K^{(3,2)}$	64	12	4096	12
$K^{(5,3)}$	196	21	38416	21
$K^{(7,4)}$	400	30	160000	30

Superconvergence and Error Reduction

$u_x + u_y + u_t = 0$, $u_0(x,y) = \sin(x + 3y)$ | $1.3u_x + 0.8u_y + u_t = 0$, $u_0(x,y) = \sin(x)\cos(y)$

Error contours (log)



Global Analysis

$u_x + u_y + u_t = 0$, $u_0(x,y) = \sin(x + 3y)$:

N	Unfiltered L^2-Error	Unfiltered Order	Filtered L^2-Error	Filtered Order
$\mathbb{P}^2$				
20	3.6-e03	-	5.0e-04	-
40	4.6e-04	2.95	1.4e-05	5.15
80	5.9e-05	2.97	4.1e-07	5.12
$\mathbb{P}^3$				
20	2.1e-04	-	3.9e-04	-
40	1.3e-05	3.95	8.2e-08	12.19
80	8.5e-07	3.98	3.9e-12	7.72

$1.3u_x + 0.8u_y + u_t = 0$, $u_0(x,y) = \sin(x)\cos(y)$:

N	Unfiltered L^2-Error	Unfiltered Order	Filtered L^2-Error	Filtered Order
$\mathbb{P}^2$				
20	3.4e-04	-	6.7e-05	-
40	1.7e-05	4.33	1.1e-06	5.92
80	2.1e-06	3.00	1.8e-08	5.97
$\mathbb{P}^3$				
20	2.6e-06	-	8.1e-06	-
40	1.6e-07	4.00	3.4e-08	7.87
80	1.0e-08	4.00	1.4e-10	7.97

Smoothness Recovery

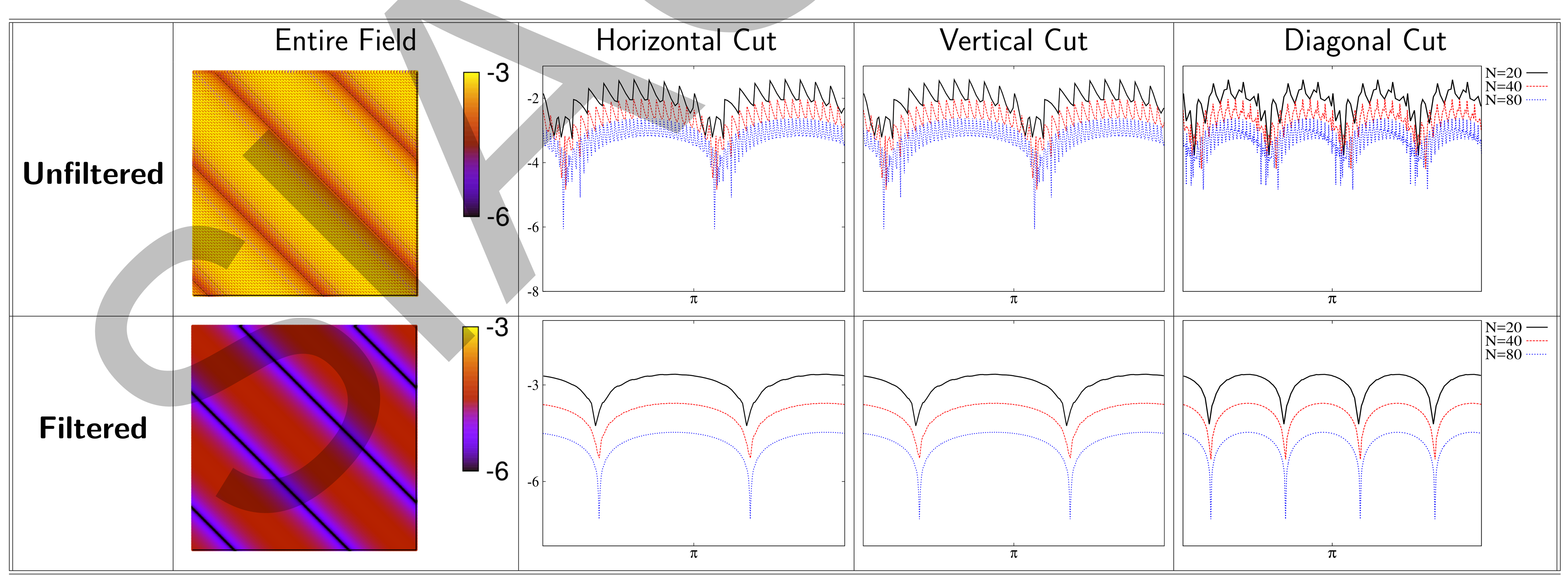
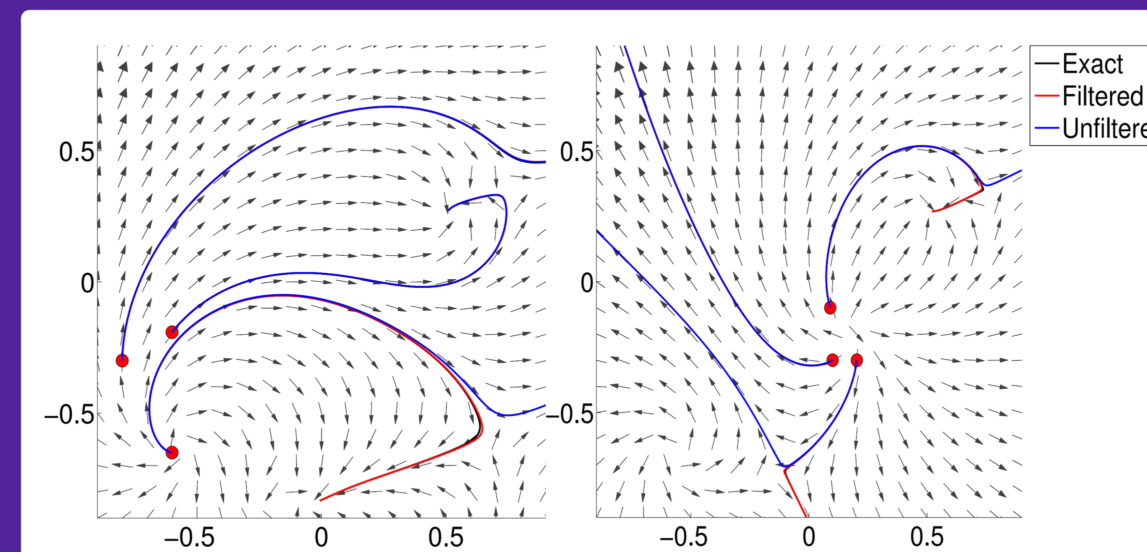


Figure: Error contours (log) for the solution to $u_x + u_y = 0$, $u_0(x,y) + \sin(x+y)$, $T = 2$ over an uniform mesh and using a $\mathbb{P}^1$ basis.

Applications to Streamline Visualisation



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[1] Cockburn, Bernardo and Luskin, Mitchell and Shu, Chi-Wang and Süli, Endre. Enhanced Accuracy by Post-Processing for Finite Element Methods for Hyperbolic Equations. *Math. Comput.*, 72(242):577–606, April 2003.

[2] J. Docampo S., J. K. Ryan, M. Mirzargar, and R. M. Kirby. Multi-dimensional filtering: Reducing the dimension through rotation. *ArXiv e-prints*, October 2016.

Conclusions

Line filtering preserves the properties of traditional tensor product filtering, including smoothness recovery and improvement in the convergence rate. Furthermore, the numerical results suggest that the filtered solution has lower error than the original one. Using this lower dimension approach, SIAC filtering becomes a computationally efficient technique for multidimensional data, allowing for applications to flow visualisation by improving the quality of the underlying field.

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